WHAT IF EUCLID OWNED AN IPAD?

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Cabrini College Colloquium October 2010

DIGITAL MATHEMATICS

Our Challenge: Teaching ancient mathematics using modern technology!

A talk in three parts:

Math inspired art and animation

Classroom Activities explored with technology

Number Theory's relationship to technology

EUCLID

Teacher, born in Alexandria Egypt about 325 BC.

Before 300 BC there are no complete math manuscripts.

Elements

About 2300 years old.

This text was the center of all mathematical teaching for over 2000 years.

Stamp originates from the Maldives Islands

EUCLID'S ELEMENTS

Definitions- Statements conveying fundamental character- for example: Points, lines and planes.

Postulates- a fundamental principle that is assumed to be true. Postulates are axioms, ie they are assumed to be true without proof.

Propositions – These are theorems. These come with proof.

EUCLID'S PYTHAGOREAN THEOREM

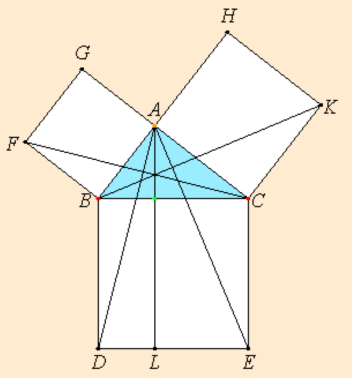
Euclid's Elements
Book I
Proposition 47

In right-angled triangles the square on the side opposite the right angle equals the sum of the squares on the sides containing the right angle.

Let *ABC* be a right-angled triangle having the angle *BAC* right.

I say that the square on *BC* equals the sum of the squares on *BA* and *AC*.

Describe the square *BDEC* on *BC*, and the squares *GB* and *HC* on *BA* and *AC*. Draw *AL* through *A* parallel to either *BD* or *CE*, and join *AD* and *FC*. I.46 I.31 Post.1



Since each of the angles *BAC* and *BAG* is right, it follows that with a straight line *BA*, and at the point *A* on it, the two straight lines *AC* and *AG* not lying on the same side make the adjacent angles equal to two right angles, therefore *CA* is in a straight line with *AG*. I.Def.22 I.14

For the same reason *BA* is also in a straight line with *AH*.

Since the angle *DBC* equals the angle *FBA*, for each is right, add the angle *ABC* to each, therefore the whole angle *DBA* equals the whole angle *FBC*. I.Def.22 Post.4 C.N.2

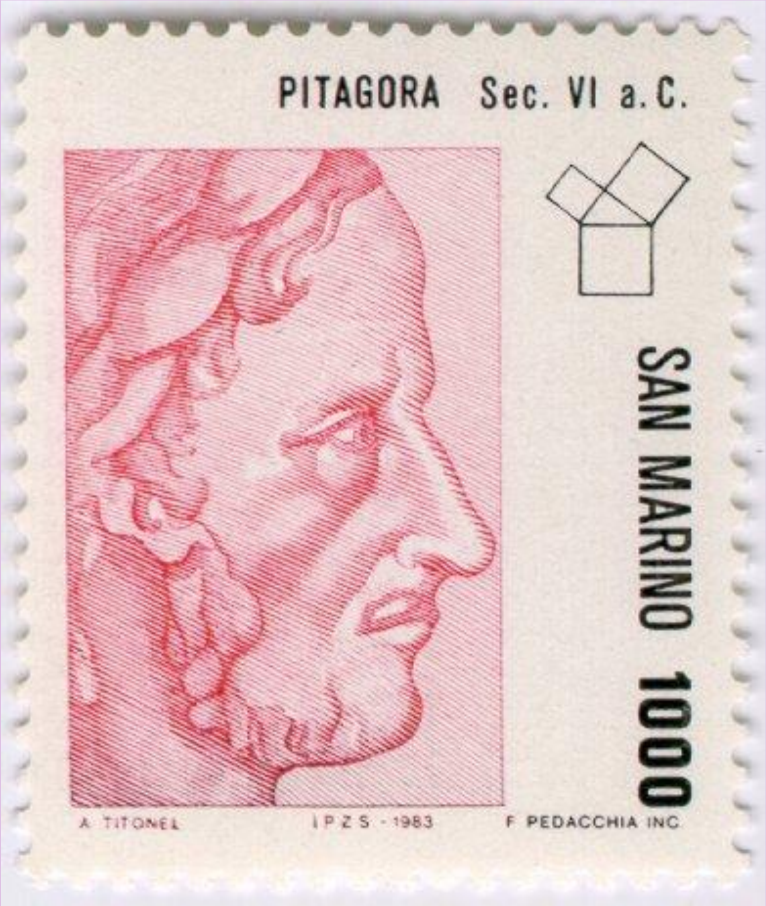
Since *DB* equals *BC*, and *FB* equals *BA*, the two sides *AB* and *BD* equal the two sides *FB* and *BC* respectively, and the angle *ABD* equals the angle *FBC*, therefore the base *AD* equals the base *FC*, and the triangle *ABD* equals the triangle *FBC*. I.Def.22 I.4

PYTHAGORAS

Ancient Greek mathematician, best known for his theorem:

Given any right triangle, the area of the square on the hypotenuse is equal to the sum of the areas of the squares on the other two sides.

$$a^2 + b^2 = c^2$$



EUCLID'S PYTHAGOREAN THEOREM

To understand Euclid's proof, we need some explanation.

EUCLID'S PYTHAGOREAN THEOREM

Let's now turn our attention to Euclid's Proof of the Pythagorean theorem.

Another Pythagorean Animation

PYTHAGOREAN SOUND BITES

Second Most Read Text

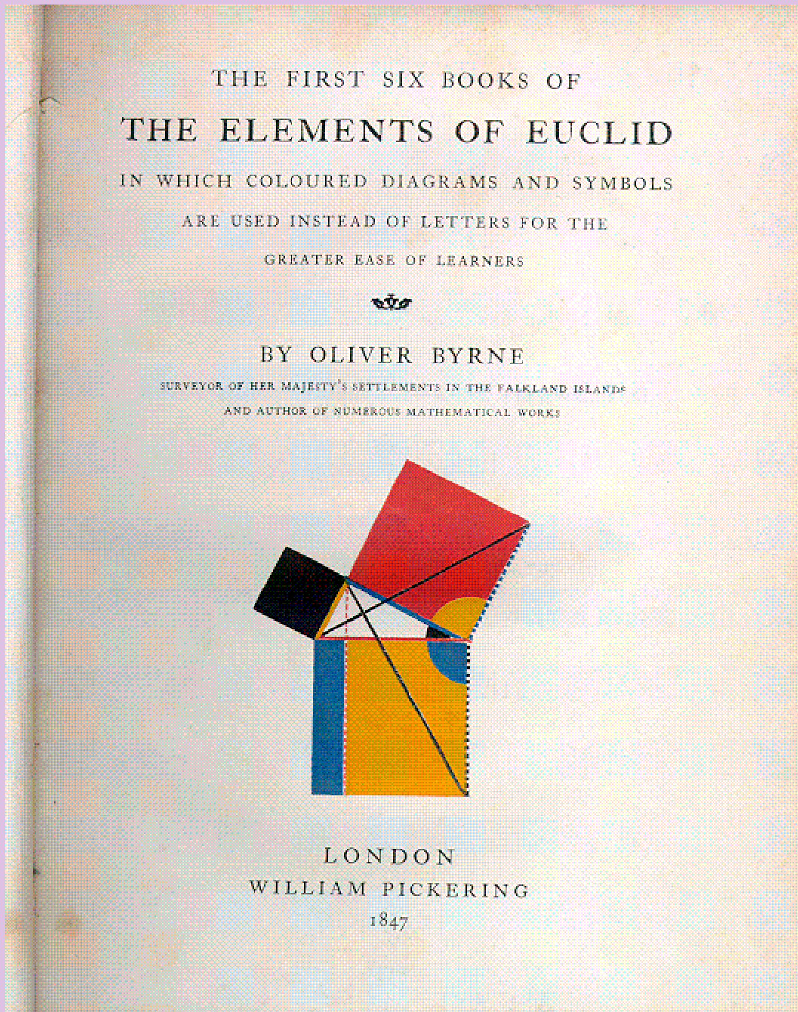
Abraham Lincoln

James Garfield

http://www.cut-the-knot.org/pythagoras/

gives a home to 88 different proofs of the Pythagorean theorem.

EUCLID FOR DUMMIES? PYTHAGOREAN SOUND BITES



THE FIRST SIX BOOKS OF

THE ELEMENTS OF EUCLID

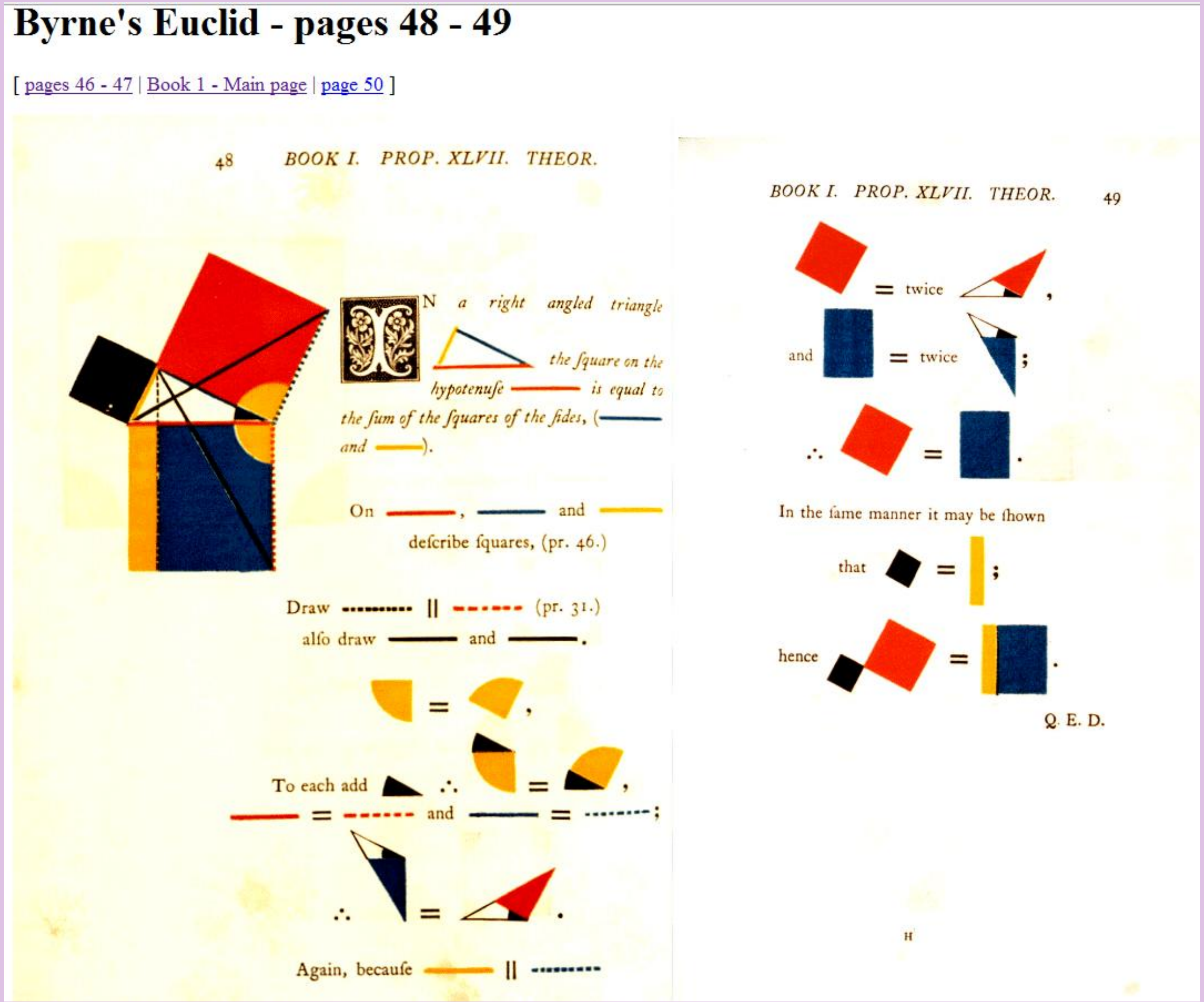
IN WHICH COLOURED DIAGRAMS AND SYMBOLS ARE USED INSTEAD OF LETTERS FOR THE GREATER EASE OF LEARNERS

BY OLIVER BYRNE

SURVEYOR OF HER MAJESTY'S SETTLEMENTS IN THE FALKLAND ISLANDS AND AUTHOR OF NUMEROUS MATHEMATICAL WORKS

LONDON
WILLIAM PICKERING
1847

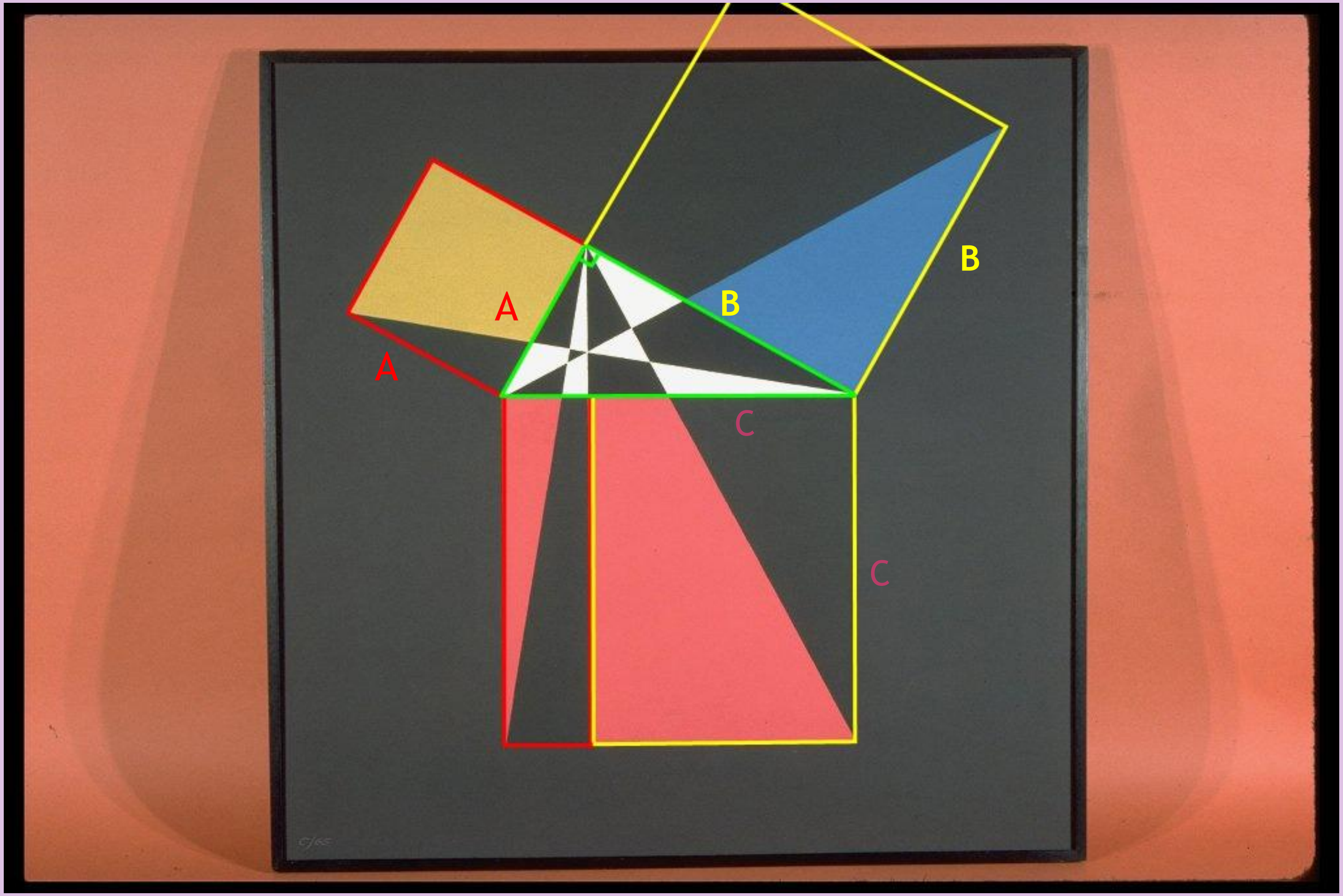
EUCLID FOR DUMMIES? PYTHAGOREAN SOUND BITES



1847

CROCKETT JOHNSON AND PYTHAGOREAN SOUND BITES

CROCKETT JOHNSON AND PYTHAGOREAN SOUND BITES



CROCKETT JOHNSON ON HIS PAINTINGS

"A decade ago upon belatedly discovering the aesthetics values in the Pythagorean right triangle and Euclidean geometry, I began a series of geometrical paintings from famous mathematical theorems, both ancient and modern. Theorems generally are universal in application and can be adapted in constructions of nearly any size and shape. The paintings were executed, as is my current work, in hard edge and flat mass, with colors focusing in intensity or contrast up the sense of the theorems."

C. Johnson, On the Mathematics of Geometry in my Abstract Paintings, *Leonardo* **5**, *1972*.

EUCLIDEAN ALGORITHM

The Greatest Common Divisor (gcd) is the largest positive integer that divides the numbers without a remainder.

Find the gcd(48, 21)

EUCLIDEAN ALGORITHM

The Euclidean Algorithm is an efficient method of calculating the greatest common divisor of two numbers.

For any pair of positive integers a and b, we may find the gcd(a,b) by repeated use of division to produce a decreasing sequence of integers $r_1 > r_2 > \ldots$ as follows:

$$a = bq_1 + r_1 \qquad 0 < r_1 < b$$

$$b = r_1 q_2 + r_2 \qquad 0 < r_2 < r_1$$

$$r_1 = r_2 q_3 + r_3 \qquad 0 < r_3 < r_2 \ldots$$

We repeat this process until we get a remainder of zero. The last non-zero remainder is the gcd(a,b).

EUCLIDEAN ALGORITHM

Find the gcd(48, 21)

48 = (21) * (2) + 6

21 = (6) * (3) + 3

6 = (3) * (2) + 0

Three is the last non-zero remainder, so gcd(48, 21)= 3.

EUCLIDEAN ALGORITHM

Find the gcd(48, 21)

In the Elements, Euclid performs these operations on lines.

Click to start and stop

EUCLIDEAN ALGORITHM

Find the gcd(3108, 1524)

3108 =(1524) * (2) + 60

1524 = (60) * (25) + 24

60 = (24) * (2) + 12

24 = 12 * (2) + 0

Twelve is the last non-zero remainder, thus gcd(3108, 1524) = 12

WEBQUEST

Online Class for Teach for American Program for students at American University

Course Description: Advanced Exploration of Secondary Mathematics. This course deepens teachers' understandings of math concepts and helps them understand the overall secondary math curriculum, as well as how to connect math concepts to curricular topics.

WEBQUEST

Topics spanned Fractions to Functions.

Challenge - Create an assignment that:

- Is relevant to curriculum
- Looked at a topic in-depth
- Provide the teachers an assignment to use with their students
- Used advantage of modern technology

WEBQUEST

Directions:

- Suggest that students work in groups.
- Notes what prior math topic exposure up to and including factoring of binomials, but not necessarily the quadratic formula or completing the square.
- Expects students to have used algebra tiles.
- Give a background story was created to grab the attention of the students.
- Include helpful websites.

Students were able to ask for help at anytime using email, online chats or virtual classroom meetings.

WEBQUEST

The problem:

You have found a room that holds many riddles. In order to leave you must solve one:

One square, and ten roots of the same, are equal to thirty-nine *dirhems*. That is to say, what must be the square which, when increased by ten of its own roots, amounts to 39?

- What is a root?
- What is a dirhem?

WEBQUEST

Abū ʿAbdallāh Muḥammad ibn Mūsā al-Khwārizmī

The Compendious Book on Calculation by Completion and Balancing (*al-Kitab al-mukhtasar fi hisab al-jabr wa'l-muqabala*

This was published in the year 825.

A dirhem is a monetary unit.

WEBQUEST

One square, and ten roots of the same, are equal to thirty-nine *dirhems*. That is to say, what must be the square which, when increased by ten of its own roots, amounts to 39?

How might you write this with modern notation?

$$x^2 + 10x = 39$$

Webquest

One square, and ten roots of the same, are equal to thirty-nine *dirhems*. That is to say, what must be the square which, when increased by ten of its own roots, amounts to 39?

$$x^2 + 10x = 39$$

How would you solve the problem?

WebQuest

One square, and ten roots of the same, are equal to thirty-nine *dirhems*. That is to say, what must be the square which, when increased by ten of its own roots, amounts to 39?

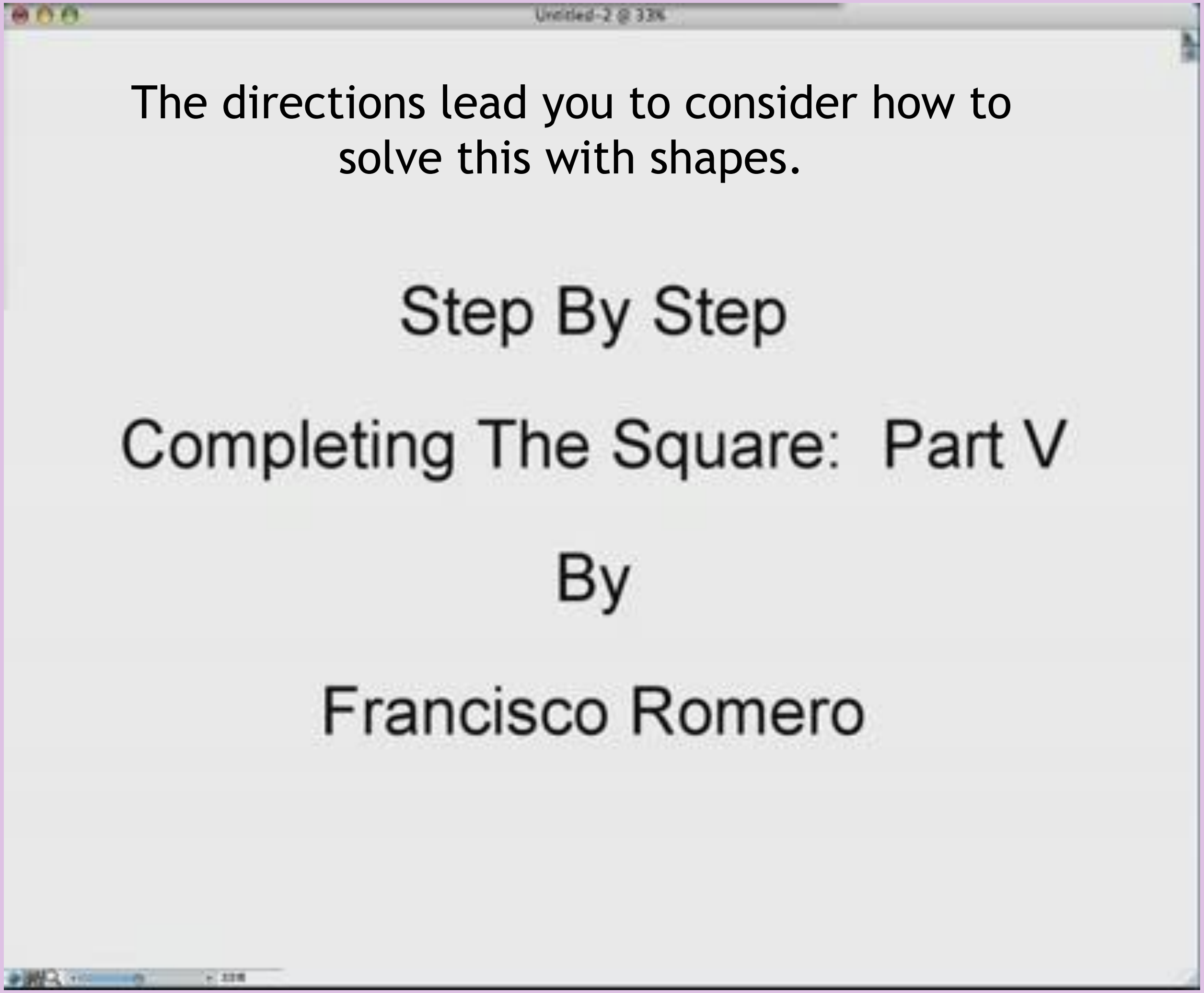
$$x^2 + 10x = 39$$

How would you solve the problem?

- Guess and Check
- Graphing calculator
- Factoring
- Quadratic formula (if known)
- Completing the square (if known)

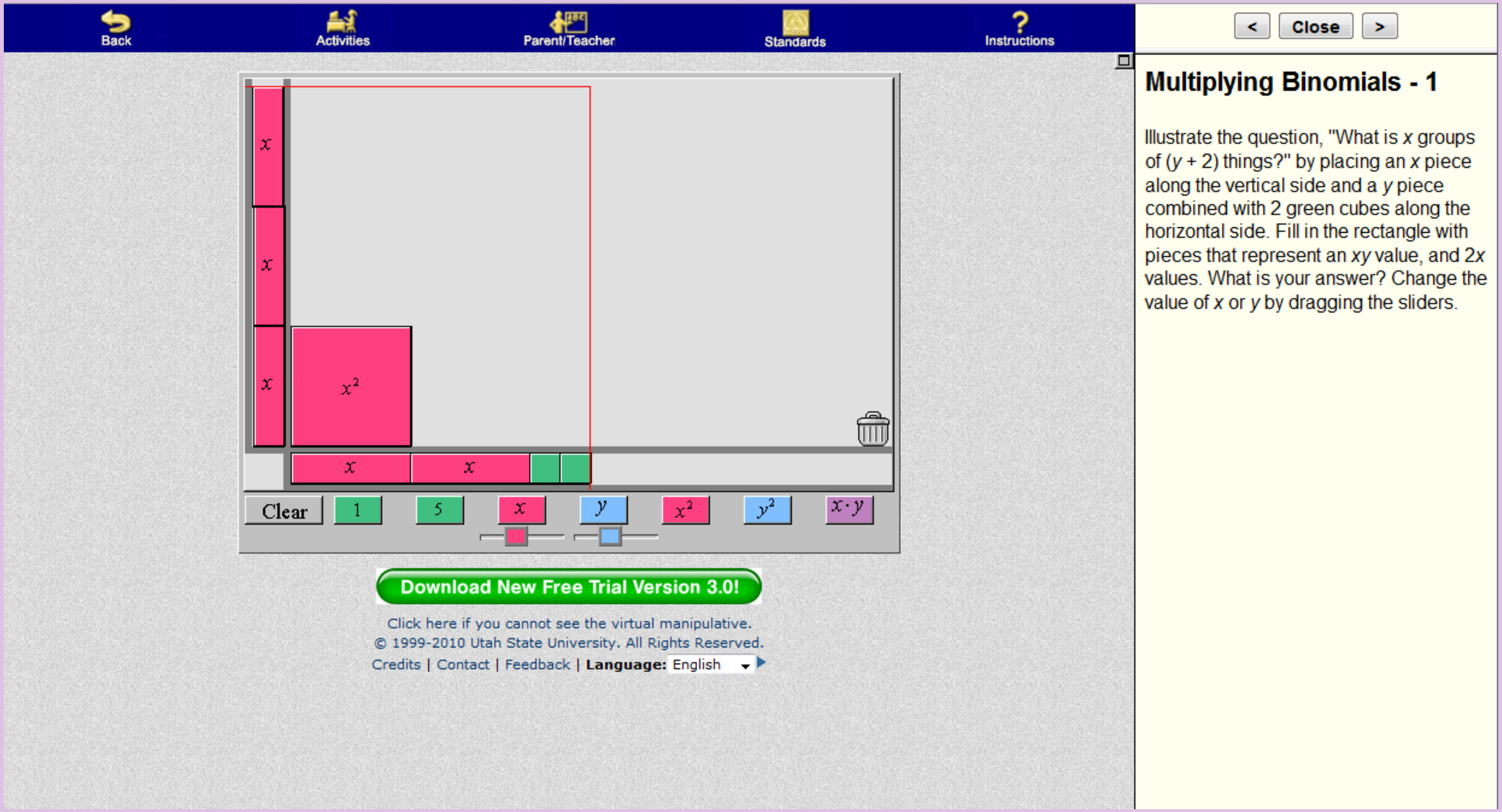
WEBQUEST

The directions lead you to consider how to solve this with shapes.



http://www.youtube.com/watch?v=Ax2mIah7bkQ&feature=channel

WEBQUEST

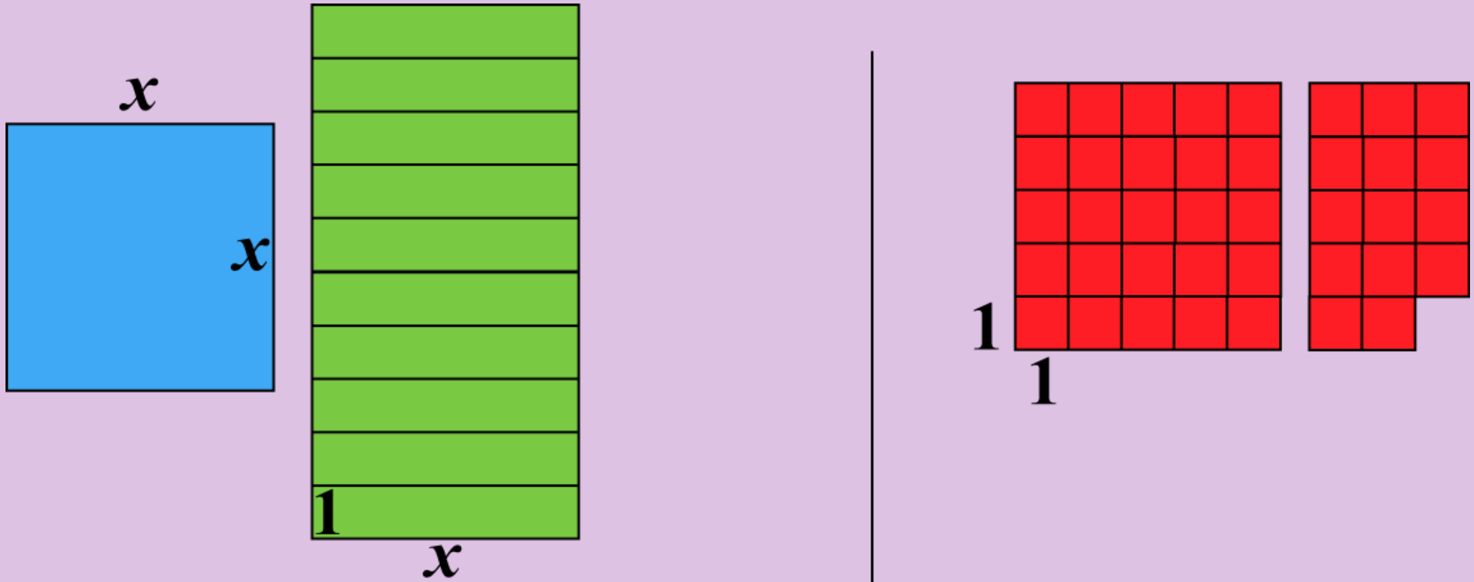


The directions lead you to use virtual manipulatives.

http://nlvm.usu.edu/en/nav/frames_asid_189_g_3_t_2.html?open=activities&from=topic_t_2.html

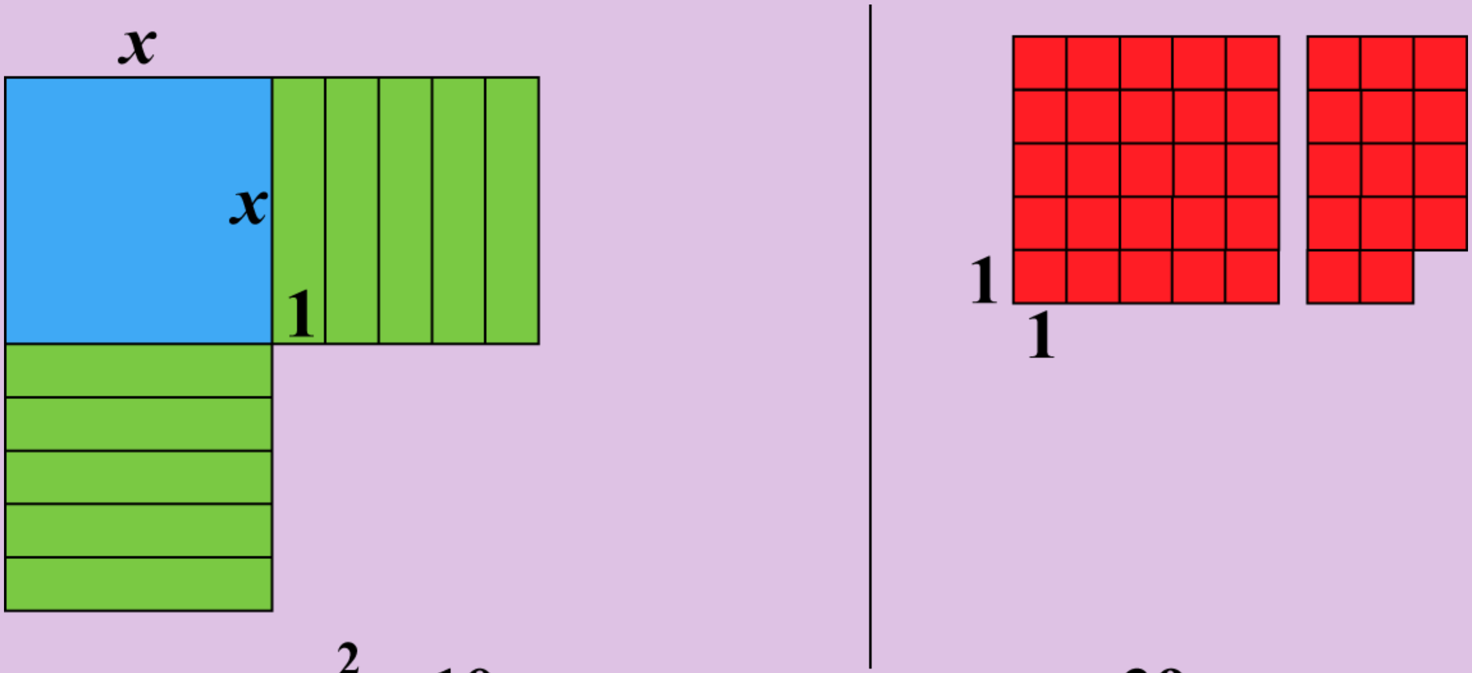
WEBQUEST

Students were asked to represent this problem with algebra tiles. The initial set up might look like:



$$x^2 + 10x = 39$$

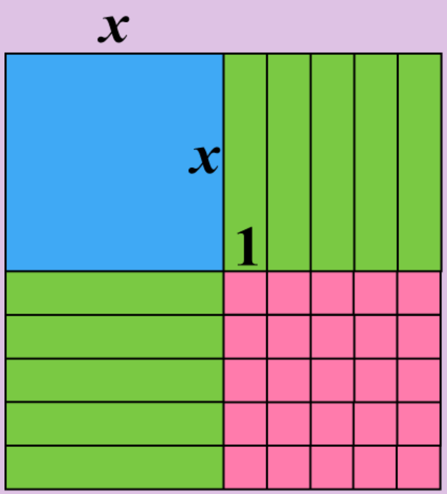
WEBQUEST

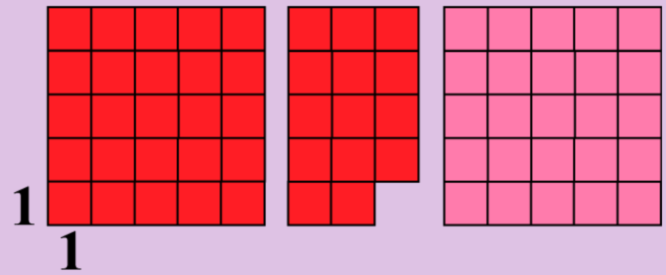


$$x^2 + 10x = 39$$

WEBQUEST

Completing the square yields:





$$x^2 + 10x + 25 = 39 + 25$$

$$(x + 5)^2 = 64$$

$$x = \{3, -13\}$$

WEBQUEST

The solution written:

You half the number of roots, which in the present instance yields five. This you multiply by itself; the product is twenty-five. Add this to thirty-nine; the sum is sixty-four. Now take the root of this, which is eight, and subtract from it half the number of roots, which is five; the remainder is three. This is the root of the square which you sought for; the square itself is nine.

WEBQUEST

Traditional algebraic approach for completing the square:

Scrap:

$$\frac{10}{2} = 5$$

$$5^2 = 25$$

Calculations:

$$x^2 + 10x = 39$$

$$x^2 + 10x + 25 = 39 + 25$$

$$(x+5)^2 = 64$$

$$x + 5 = \pm 8$$

$$x = \{3, -13\}$$

WEBQUEST

al-Khwārizmī classifies linear and quadratic equations in six forms, with solutions justified geometrically.

The six cases are:

Squares equal to roots	$x^2 = 9x$
Squares equal to numbers	$x^2 = 9$
Roots equal to numbers	$x = 9$
Squares and roots equal to numbers	$x^2 + x = 6$
Squares and numbers equal to roots	$x^2 + 4 = 5x$
Roots and numbers equal to squares	$x^2 = 4 + 3x$

WEBQUEST

The Quadratic Formula

$$ax^2 + bx + c = 0$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

WEBQUEST

Outcomes:

- Students liked the assignment.
- Underscored the connections between algebra and geometry.
- Allowed for differentiated learning in the classroom
- Took advantage of technology.
- Asked them to look for other appropriate digital material that could be applicable.

QUADRATICS

Evidence of working quadratic equations in other cultures:

- Babylonians- Clay tablets (400 BC)
- Chinese-*Nine Chapters of Mathematical Art* (100 BC)
- Greeks
  - Euclid's *Elements*
  - Apollonius (262 -190 BC) - *The Conics*
  - Diophantus (200-284 AD) – *Arithmetica*

http://aleph0.clarku.edu/~djoyce/java/elements/bookII/propII4.html

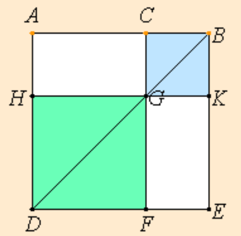
QUADRATICS

Euclid's Elements
Book II
Proposition 4

If a straight line is cut at random, then the square on the whole equals the sum of the squares on the segments plus twice the rectangle contained by the segments.

Let the straight line *AB* be cut at random at *C*.

I say that the square on *AB* equals the sum of the squares on *AC* and *CB* plus twice the rectangle *AC* by *CB*.



Describe the square *ADEB* on *AB*. Join *BD*. Draw *CF* through *C* parallel to either *AD* or *EB*, and draw *HK* through *G* parallel to either *AB* or *DE*. I.46 I.31

Then, since *CF* is parallel to *AD*, and *BD* falls on them, the exterior angle *CGB* equals the interior and opposite angle *ADB*. I.29

But the angle *ADB* equals the angle *ABD*, since the side *BA* also equals *AD*. Therefore the angle *CGB* also equals the angle *GBC*, so that the side *BC* also equals the side *CG*. I.5 I.6

But *CB* equals *GK*, and *CG* to *KB*. Therefore *GK* also equals *KB*. Therefore *CGKB* is equilateral. I.34

I say next that it is also right-angled.

Since *CG* is parallel to *BK*, the sum of the angles *KBC* and *GCB* equals two right angles. I.29

But the angle *KBC* is right. Therefore the angle *BCG* is also right, so that the opposite angles *CGK* and *GKB* are also right. I.34

Therefore *CGKB* is right-angled, and it was also proved equilateral, therefore it is a square, and it is described on *CB*.

For the same reason *HF* is also a square, and it is described on *HG*, that is *AC*. Therefore the squares *HF* and *KC* are the squares on *AC* and *CB*. I.34

Now, since *AG* equals *GE*, and *AG* is the rectangle *AC* by *CB*, for *GC* equals *CB*, therefore *GE* also equals the rectangle *AC* by *CB*. Therefore the sum of *AG* and *GE* equals twice the rectangle *AC* by *CB*. I.43

But the squares *HF* and *CK* are also the squares on *AC* and *CB*, therefore the sum of the four figures *HF*, *CK*, *AG*, and *GE* equals the sum of the squares on *AC* and *CB* plus twice the rectangle *AC* by *CB*.

But *HF*, *CK*, *AG*, and *GE* are the whole *ADEB*, which is the square on *AB*.

Therefore the square on *AB* equals the the sum of the squares on *AC* and *CB* plus twice the rectangle *AC* by *CB*.

QUADRATICS

If a straight line is cut at random, then the square on the whole equals the sum of the squares on the segments plus twice the rectangle contained by the segments.

QUADRATICS

A geometric proof:

Perfect Numbers

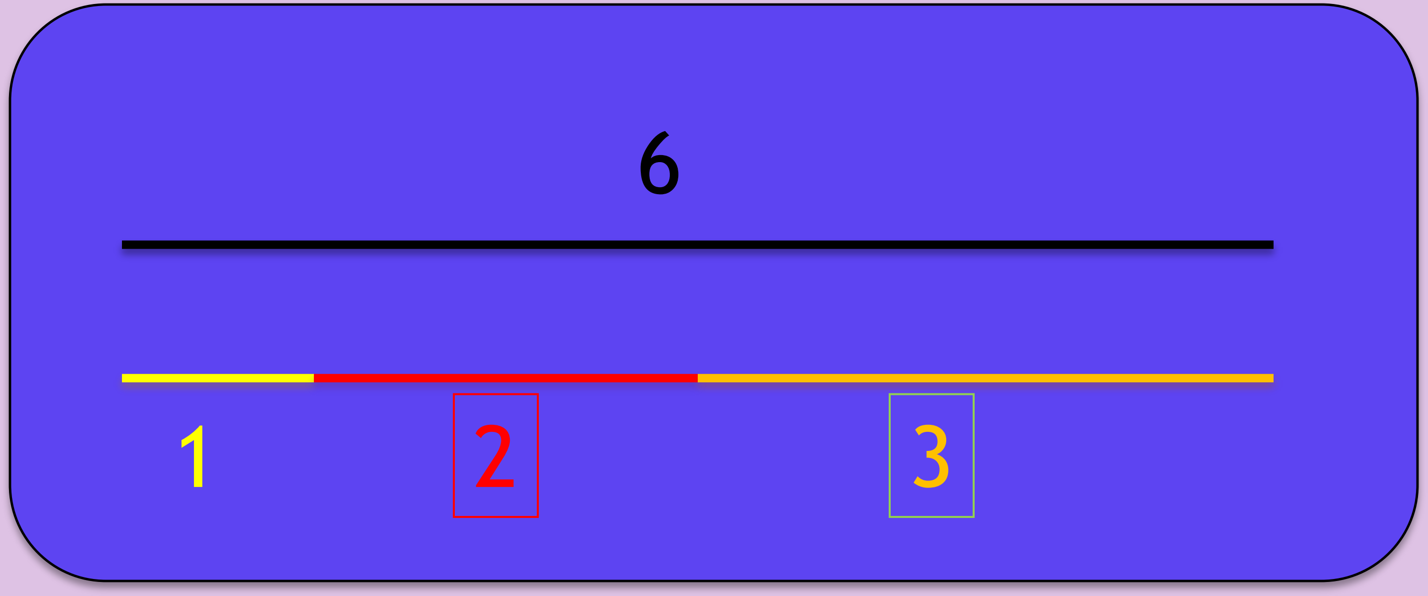
A perfect number is a positive integer that is equal to the sum of its proper divisors.

For example: The proper divisors of the number 6 are 1, 2 and 3.

$$1+2+3 = 6$$

PERFECT NUMBERS

Graphically, Euclid would represent number quantities as lines:



Perfect Numbers

What is the next perfect number?

The next perfect number is 28.

$$1 + 2 + 4 + 7 + 14 = 28$$

The first four perfect numbers are:

6, 28, 496 and 8128

PERFECT NUMBERS

Euclid's *Elements* states in Book IX, Proposition 36 :

"If as many numbers as we please beginning from a unit be set out continuously in double proportion, until the sum of all becomes a prime, and if the sum multiplied into the last make some number, the product will be perfect."

PERFECT NUMBERS

Perfect numbers can be factored into:

$$\left(2^p - 1\right) * 2^{p-1}$$

So consider values of p

p	$\left(2^p - 1\right) * 2^{p-1}$
1	$\left(2^1 - 1\right) * 2^{1-1} = 1(1) = 1$
2	$\left(2^2 - 1\right) * 2^{2-1} = 3(2) = 6$
3	$\left(2^3 - 1\right) * 2^{3-1} = 7(4) = 28$
4	$\left(2^4 - 1\right) * 2^{4-1} = 15(8) = 120$
5	$\left(2^5 - 1\right) * 2^{5-1} = 31(16) = 496$
6	$\left(2^6 - 1\right) * 2^{6-1} = 63(32) = 2016$
7	$\left(2^7 - 1\right) * 2^{7-1} = 127(64) = 8128$

Observations?

PERFECT NUMBERS

p	$\left(2^p - 1\right) * 2^{p-1}$	$\left(2^p - 1\right)$
1	$\left(2^1 - 1\right) * 2^{1-1} = 1(1) = 1$	1
2	$\left(2^2 - 1\right) * 2^{2-1} = 3(2) = 6$	3
3	$\left(2^3 - 1\right) * 2^{3-1} = 7(4) = 28$	7
4	$\left(2^4 - 1\right) * 2^{4-1} = 15(8) = 120$	15
5	$\left(2^5 - 1\right) * 2^{5-1} = 31(16) = 496$	31
6	$\left(2^6 - 1\right) * 2^{6-1} = 63(32) = 2016$	63
7	$\left(2^7 - 1\right) * 2^{7-1} = 127(64) = 8128$	127

If $\left(2^p - 1\right)$ is prime, then $\left(2^p - 1\right) * 2^{p-1}$ is perfect.

PRIMES

Prime numbers of the form

$$2^p - 1$$

are known as Mersenne Primes, honor of Marin Mersenne, an Order of the Minims monk, who studied mathematics in the 17th century.

Primes

The search for primes has become such an interest that www.mersenne.org has set up the GIMPS project.

GIMPS: Great Internet Mersenne Prime Search

- Provides a program that you can download that looks for Mersenne Primes while your computer runs.
- Imagine millions of peoples all working to find the next prime.
- Cash Incentive.

PRIMES

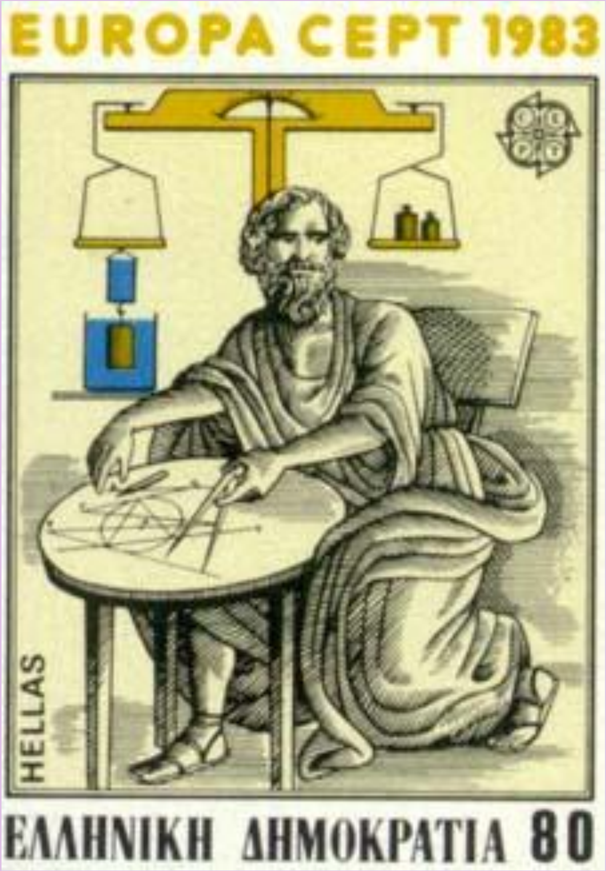
Primes numbers are used in:

Testing of computer hardware in quality control.

Public Key encryption: In this a message is encoded, locked with one key, and opened with a different key.

The one most commonly used today is the RSA Algorithm, named from the inventors Rivest, Shamir and Adleman, and it uses prime numbers to generate the keys for public key encrytion.

ARCHIMEDES



Archimedes 287-212 BC

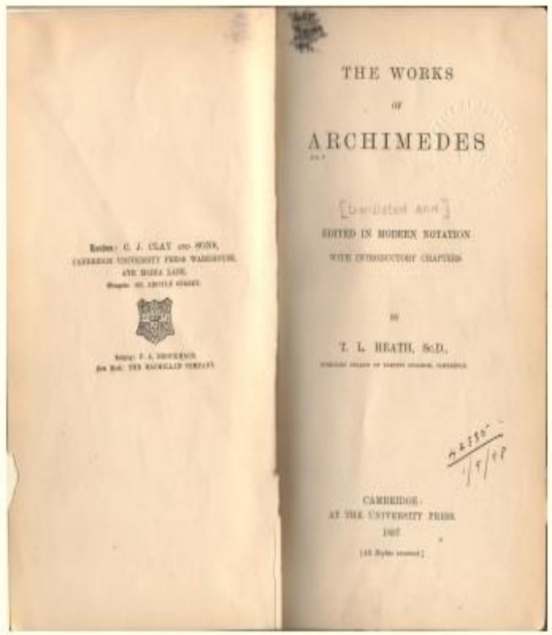
Inventor

The Quadrature of the Parabola discusses 24 propositions regarding the underlying nature of parabolic segments.

Quadrature – construction of a square that has the same area of a curved shape.

Parabolic Segment is the region bounded by a parabola and a line.

ARCHIMEDES



The images here have been taken from T. L. Heath's translation into English of Archimedes' collected works, published in 1897 by Cambridge University Press. Heath's edition is based in turn on the definitive Greek edition of J. L.Heiberg.

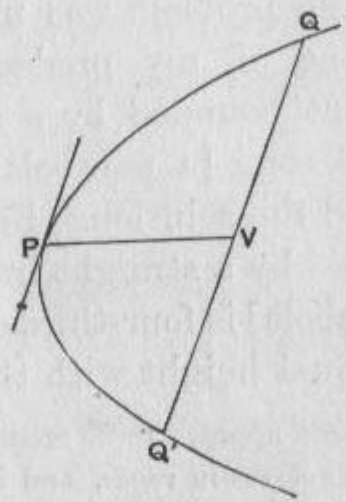
ARCHIMEDES

Proposition 1.

If from a point on a parabola a straight line be drawn which is either itself the axis or parallel to the axis, as PV, and if QQ' be a chord parallel to the tangent to the parabola at P and meeting PV in V, then

$$QV = VQ'.$$

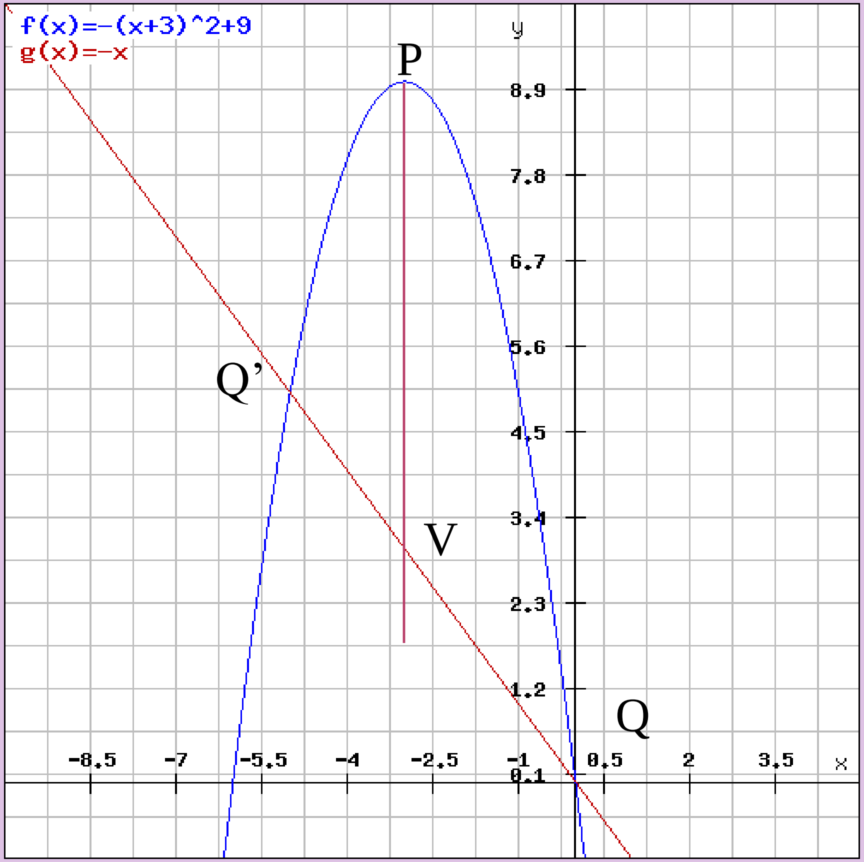
Conversely, *if $QV = VQ'$, the chord QQ' will be parallel to the tangent at P.*



* The Greek of this passage is: συμβαίνει δὲ τῶν προειρημένων θεωρημάτων ἕκαστον μηδὲν ἧσσον τῶν ἄνευ τούτου τοῦ λήμματος ἀποδεδειγμένων πεπιστευκέναι. Here it would seem that πεπιστευκέναι must be wrong and that the passive should have been used.

$$f(x) = -(x+3)^2 + 9, \quad -5 \le x \le 0$$

ARCHIMEDES



Analysis:

Q = (0, 0)

Q' = (-5, 5)

Line QQ': $y = -x$

$f'(x) = -2x - 6$

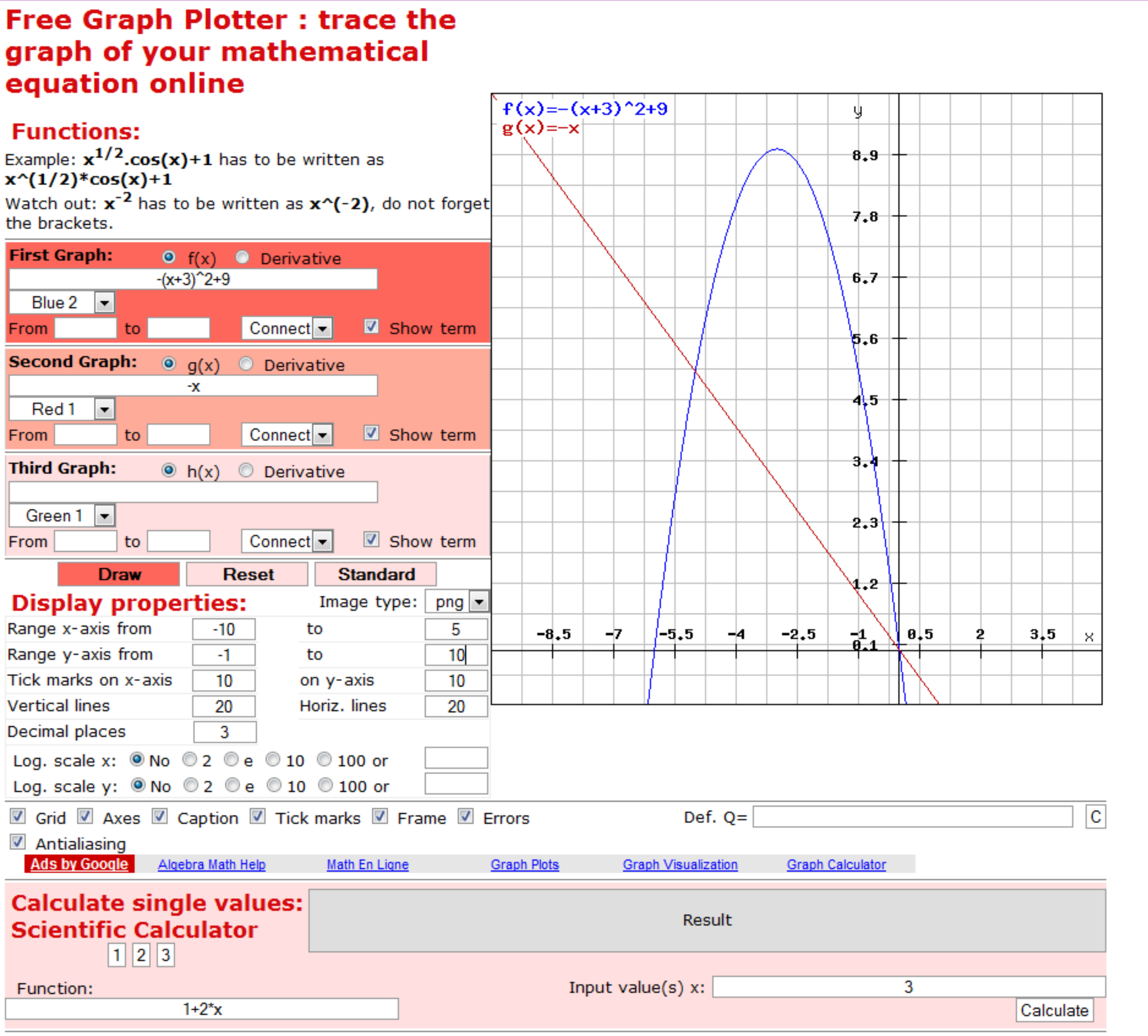
Solving $-2x - 6 = -1$, $x = -2.5$.

P = (-2.5, 8.75)

P and V share the same x coordinate V = (-2.5, 2.5)

$d(QV) = d(Q'V) = 2.5\sqrt{2}$

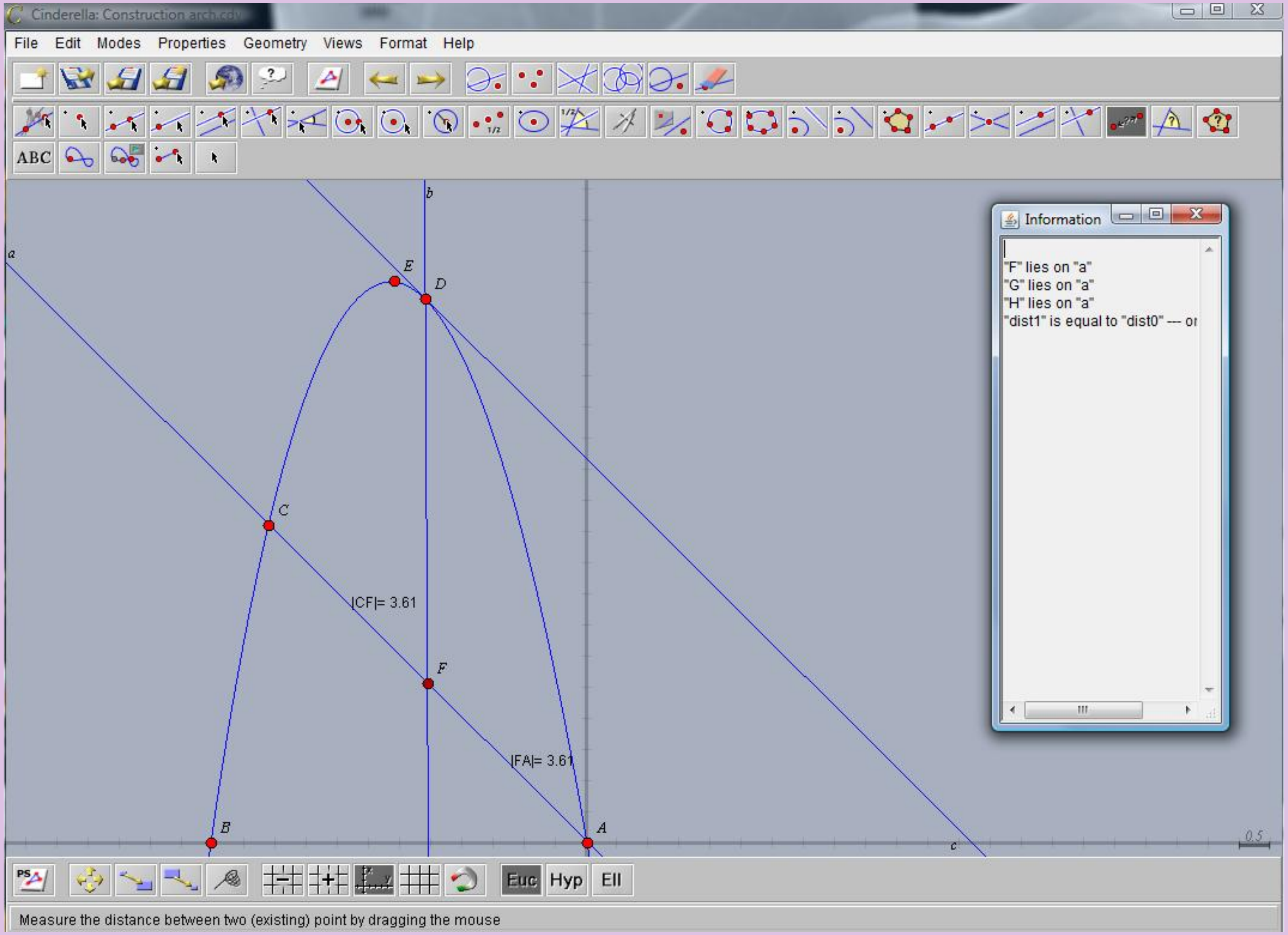
ARCHIMEDES



Free Graph Plotter : trace the graph of your mathematical equation online
Functions:
Example: x^1/2.cos(x)+1 has to be written as x^(1/2)*cos(x)+1
Watch out: x^-2 has to be written as x^(-2), do not forget the brackets.
First Graph: f(x) Derivative
-(x+3)^2+9
Blue 2
From to Connect Show term
Second Graph: g(x) Derivative
-x
Red 1
From to Connect Show term
Third Graph: h(x) Derivative
Green 1
From to Connect Show term
Draw
Reset
Standard
Display properties:
Image type: png
Range x-axis from -10 to 5
Range y-axis from -1 to 10
Tick marks on x-axis 10 on y-axis 10
Vertical lines 20 Horiz. lines 20
Decimal places 3
Log. scale x: No 2 e 10 100 or
Log. scale y: No 2 e 10 100 or
Grid Axes Caption Tick marks Frame Errors
Def. Q=
C
Antialiasing
Ads by Google
Algebra Math Help
Math En Ligne
Graph Plots
Graph Visualization
Graph Calculator
Calculate single values: Scientific Calculator
1 2 3
Result
Function:
1+2*x
Input value(s) x:
3
Calculate
f(x)=-(x+3)^2+9
g(x)=-x
y
8.9
7.8
6.7
5.6
4.5
3.4
2.3
1.2
0.1
-8.5
-7
-5.5
-4
-2.5
-1
0.5
2
3.5
x

http://www.cinderella.de/tiki-index.php

ARCHIMEDES



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Acknowledgements

- The Crockett Johnson Homepage
  - http://www.ksu.edu/english/nelp/purple/

- Peggy Kidwell
  - Director of the Mathematics Collection
  - National Museum of American History
  - We wish to thank the Smithsonian Institution for allowing us to use digital images of Crockett Johnson’s paintings.

THANKS!

- STOP